Space-Time Noncommutative
Field Theories And Unitarity

Jaume Gomis and Thomas Mehen

Department of Physics

California Institute of Technology

Pasadena, CA 91125

and

Caltech-USC Center for Theoretical Physics

University of Southern California

Los Angeles, CA 90089

`gomis, mehen@theory.caltech.edu`

We study the perturbative unitarity of noncommutative scalar field theories. Field theories with space-time noncommutativity do not have a unitary S-matrix. Field theories with only space noncommutativity are perturbatively unitary. This can be understood from string theory, since space noncommutative field theories describe a low energy limit of string theory in a background magnetic field. On the other hand, there is no regime in which space-time noncommutative field theory is an appropriate description of string theory. Whenever space-time noncommutative field theory becomes relevant massive open string states cannot be neglected.

May 2000

1. Introduction

Noncommutative field theories are constructed from conventional (commutative) field theories by replacing in the Lagrangian the usual multiplication of fields with the $\star$-product of fields. The $\star$-product is defined in terms of a real antisymmetric matrix $\theta^{\mu\nu}$ that parameterizes the noncommutativity of Minkowski space-time[1]

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \qquad \mu, \nu = 0, \ldots, D-1. \tag{1.1}$$

The $\star$-product of two fields $\phi_1(x)$ and $\phi_2(x)$ is given by

$$\phi_1(x) \star \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}}\phi_1(x+\alpha)\,\phi_2(x+\beta)|_{\alpha=\beta=0}. \tag{1.2}$$

The noncommutativity in (1.1) gives rise to a space-time uncertainty relation

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}| \tag{1.3}$$

which leads to number of unusual phenomena such as the mixing of the ultraviolet with the infrared as well as apparent acausal behavior [1]-[7].

These field theories are non-local and this nonlocality has important consequences for the dynamics [1]-[7]. The structure of the product in (1.2) leads to terms in the action with an infinite number of derivatives of fields which casts some doubts on the unitarity of noncommutative field theories. In this paper we will check the unitarity of scalar noncommutative field theories at the one loop level and show that theories with $\theta^{0i} = 0$ are unitary while theories with $\theta^{0i} \neq 0$ are not unitary.

Noncommutative field theories with space noncommutativity (that is $\theta^{0i} = 0$) have an elegant embedding in string theory [8][9][10]. They describe the low energy excitations of a D-brane in the presence of a background magnetic field[2]. In this limit [10] the relevant description of the dynamics is in terms of the noncommutative field theory of the massless open strings. Both the massive open strings and the closed strings decouple[3]. The consistent truncation of the full unitary string theory to field theory with space noncommutativity leads one to suspect that these field theories are unitary. Moreover, these

---

[1] Throughout the paper we will use the $(+, -, \ldots, -)$ convention for the signature of space-time.

[2] We will sometimes refer to these theories as magnetic theories.

[3] In [11]-[15] one-loop string theory amplitudes were shown to exhibit this decoupling. See also [16]-[17].

field theories are nonlocal in space but are local in time. Therefore, a Hamiltonian can be constructed and it gives rise to unitary time evolution of noncommutative magnetic field theories.

Theories with space-time noncommutativity[4] (that is $\theta^{0i} \neq 0$) have an infinite number of time derivatives of fields in the Lagrangian and are nonlocal in time. The commutator in (1.1) leads to noncommutativity of the time coordinate. Noncommutativity of the time coordinate and the corresponding nonlocality in time results in theories where it is far from clear whether the usual framework of quantum mechanics makes sense. As such, noncommutative field theories with space-time noncommutativity are excellent laboratories in which to test the possible breakdown of the conventional notion of time or the familiar framework of quantum mechanics in string theory at the Planck scale[5]. In this paper we will test in these exotic field theories one of the basic principles of quantum mechanics, the existence of a unitary S-matrix. We explicitly show that several one loop amplitudes in noncommutative scalar electric field theory are not unitary which demonstrates that noncommutative field theories with space-time noncommutativity clash with quantum mechanics.

This field theory result meshes very nicely with string theory expectations. $\theta^{0i} \neq 0$ is obtained by studying string theory in the presence of a background electric field (recent work in this direction has recently appeared in [20]-[22], see also [23]). There are three important parameters that characterize the open strings [24][25][10]: α', the metric $G_{\mu\nu}$ and the noncommutativity matrix $\theta^{\mu\nu}$. One must also keep in mind that there is an upper critical value on the magnitude of the background electric field E_c [26]-[28], beyond this value the string vacuum becomes unstable. It can be shown [20][21][22] using the relation between these open string parameters with the closed string metric and background electric field that it is impossible to take a consistent limit of string theory in which $\theta^{\mu\nu}$ and $G_{\mu\nu}$ are kept fixed while $\alpha' \to 0$. Therefore, unlike the case of strings in a background magnetic field, it is impossible to find a limit of string theory in which one is left only with a noncommutative field theory with fixed background metric $G_{\mu\nu}$ and space-time noncommutativity parameter θ^{0i}. It is possible to find a limit of string theory [20][21]

---

[4] Likewise, we will sometimes refer to these theories as electric theories.

[5] In recent years, several examples of the breakdown of the conventional notion of space at very short distances have been found in string theory such as in topology changing transitions [18][19]. It seems, therefore, imperative to address the issue of possible breakdown of time.

with nonvanishing θ^{0i} in which the closed strings decouple. However, $\theta^{0i} \sim \alpha'$ making it impossible to decouple massive open string states and keep θ^{0i} finite. Thus, there is no sense in which the electric field theories give an approximate description of a limit of string theory. The lack of decoupling of the massless open string modes from the massive ones gives very strong indication that the noncommutative field theory truncation to the massless modes is not unitary. This is consistent with what we find from our field theory analysis.

The paper is organized as follows. In section 2 we compute several one loop amplitudes in noncommutative scalar field theory and show that Feynman diagrams of space-time noncommutative theories do not satisfy the usual cutting rules and the S-matrix does not satisfy unitarity constraints. We also show that one loop amplitudes are unitary in the presence of only space noncommutativity and the Feynman diagrams satisfy the cutting rules. We conclude in section 3 with a discussion of our results and their relation to limits of string theory in electromagnetic field backgrounds.

2. Unitarity of Noncommutative Scalar Field Theory

In this section we examine one loop diagrams of noncommutative ϕ^3 and ϕ^4 theories to see if they satisfy constraints from unitarity. For on-shell matrix elements unitarity implies that

$$2 \operatorname{Im} M_{ab} = \sum_n M_{an} M_{nb} \tag{2.1}$$

where M_{ab} is the transition matrix element between states a and b. The sum over intermediate states on the right hand side includes phase space integrations for each particle in n. Quantum field theories actually satisfy more restrictive relations called generalized unitarity relations or cutting rules. These state that the imaginary part of a Feynman diagram can be obtained by the following procedure: First, "cut" the diagram by drawing a line through virtual lines such that the graph is severed in two. Next, wherever the cut intersects a virtual line, place that virtual particle on-shell by replacing the propagator with a delta function:

$$\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow -2\pi i \, \delta(p^2 - m^2). \tag{2.2}$$

Summing over all cuts yields the imaginary part of the Feynman diagram. Cutting rules are a generalization of (2.1) to Feynman diagrams. Unitarity of the S-matrix (2.1) follows

3

from the cutting rules[6]. Note that the cutting rules are more restrictive than the constraint of unitarity since they apply to off-shell Green's functions as well as S-matrix elements.

We will first show that the two-point function of the noncommutative ϕ^3 theory does not obey the usual cutting rules when there is space-time noncommutativity ($\theta^{0i} \neq 0$). In the case of space noncommutativity ($\theta^{0i} = 0$, $\theta^{ij} \neq 0$) the cutting rules are satisfied. Next, we consider $2 \rightarrow 2$ scattering in noncommutative ϕ^4 theory. The S-matrix is nonunitary at one-loop if $\theta^{0i} \neq 0$, but is unitary if the noncommutativity is only in the spatial directions.

It is somewhat surprising that Feynman diagrams of space-time noncommutative theories do not obey the usual cutting rules. Since the Feynman rules for the vertices of noncommutative theories are manifestly real functions of momenta, one would expect that Feynman graphs could only develop a branch cut when internal lines go on-shell. This would imply that the imaginary parts of Feynman diagrams would be given by the same cutting rules as ordinary commutative field theories. The resolution of this puzzle requires an examination of the high energy behavior of the oscillatory factors that typically arise in these theories. We will find that a necessary condition for one-loop Feynman integrals to converge in these theories is that the following inner product

$$p \circ p = -p^\mu \theta^2_{\mu\nu} p^\nu, \tag{2.3}$$

be positive definite. This inner product is positive definite when p_μ and $\theta_{\mu\nu}$ are analytically continued to Euclidean space. In Minkowski space $p \circ p$ can be negative if $\theta^{0i} \neq 0$. Feynman integrals can be defined via analytic continuation, but the resulting amplitudes will develop branch cuts when $p \circ p < 0$. These additional branch cuts are responsible for the failure of cutting rules and unitarity in noncommutative theories with space-time noncommutativity.

For $p \circ p = 0$, the S-matrix does not suffer from lack of unitarity, but is ill-defined because of infrared divergences. $p \circ p = 0$ is possible whether the noncommutativity is space-time or space-space. Obviously an outstanding problem in noncommutative field theory is to construct the infrared safe observables of the theory. This may require all order resummation of infrared divergent terms in the perturbative series. We will not atttempt to address this issue in this paper, and focus only on perturbative unitarity constraints for matrix elements which do not suffer from infrared singularities.

---

[6] This assumes of course that the poles of the propagators correspond to physical states. In gauge theories unphysical states can propagate in loops and one must demonstrate that these states decouple from the physical S-matrix. This will not be a concern for the scalar theories considered in this paper.
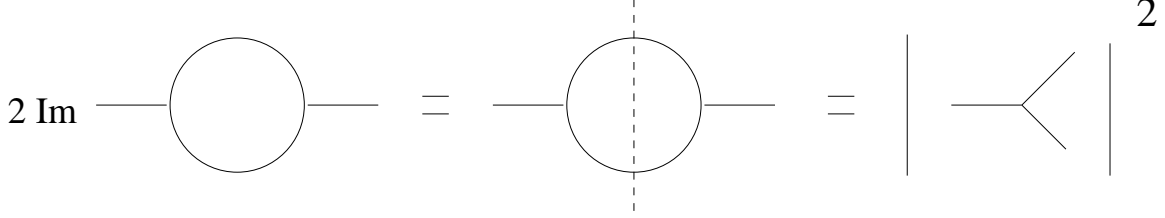
2.1. Noncommutative ϕ^3 two-point function

Fig. 1: Generalized unitarity relation for ϕ^3 two-point function.

The cutting rule for the noncommutative ϕ^3 theory two-point function at lowest order is displayed in fig. 1. The propagators of fields in noncommutative field theories are identical to those of commutative field theory. The Feynman rule for the vertex in this theory is

$$-i\,\lambda \cos\left(\frac{k \wedge q}{2}\right), \quad k \wedge q = k_\mu \theta^{\mu\nu} q_\nu. \tag{2.4}$$

where k and q are any two of the momenta flowing into the vertex. Because of conservation of momentum and the antisymmetry of $\theta_{\mu\nu}$ it does not matter which two momenta are chosen. The amplitude for the one loop diagram appearing in fig. 1 is:

$$iM = \frac{\lambda^2}{2} \int \frac{d^D l}{(2\pi)^D} \frac{1 + \cos(p \wedge l)}{2} \frac{1}{l^2 - m^2 + i\epsilon} \frac{1}{(l+p)^2 - m^2 + i\epsilon}, \tag{2.5}$$

while the expression for the right hand side of fig. 1 is

$$\sum |M|^2 = \frac{\lambda^2}{2} \frac{1}{(2\pi)^{D-2}} \int \frac{d^{D-1}k}{2k_0} \frac{d^{D-1}q}{2q_0} \delta^D(p - k - q) \frac{1 + \cos(p \wedge k)}{2}. \tag{2.6}$$

The mass of the ϕ quanta is m and p denotes the external momentum which is not required to be on-shell. In both (2.5) and (2.6) we have used the identity $\cos^2 x = (1 + \cos(2x))/2$ to separate the planar and nonplanar contributions [29]-[31]. We will focus on the nonplanar terms since it is obvious that the planar parts satisfy unitarity constraints.

First we compute the one loop graph. We combine denominators using Feynman parameters then represent propagators via Schwinger parameters to obtain

$$M = \frac{\lambda^2}{8} \int \frac{d^D l_E}{(2\pi)^D} \int_0^1 dx \int_0^\infty d\alpha\, \alpha \left(\exp(-\alpha(l_E^2 + x(1-x)p_E^2 + m^2 - i\epsilon) + il_E \wedge p_E) + c.c.\right). \tag{2.7}$$

We have performed the usual analytic continuation $l^0 = il_E^0, p^0 = ip_E^0$. The subscript E denotes Euclidean momenta. In addition, if there is space-time noncommutativity we must analytically continue $\theta^{0i} \to -i\,\theta^{0i}$. In the string theory realization this can be easily undertood since θ^{0i} is related to a background electric field. This continuation leaves the Moyal phase invariant. Otherwise the phases appearing in (2.7), $\exp(\pm il_E \wedge p_E)$, which render the integral finite, become $\exp(\pm l_E \wedge p_E)$ and the integral is no longer convergent. Integrating over the loop momentum l_E gives

$$M = \frac{\lambda^2}{4}\frac{1}{(4\pi)^{D/2}}\int_0^1 dx \int_0^\infty d\alpha\,\alpha^{1-D/2}\exp\left(-\alpha(x(1-x)p_E^2 + m^2 - i\epsilon) - \frac{p_E \circ p_E}{4\alpha}\right). \tag{2.8}$$

We will now evaluate this integral for $D = 3$ and $D = 4$ space-time dimensions and analytically continue back the answer to Minkowski space. The amplitudes are given by

$$M_{D=3} = \frac{\lambda^2}{32\pi}\int_0^1 dx\,\frac{\exp(-\sqrt{p \circ p(m^2 - p^2 x(1-x) - i\epsilon)})}{\sqrt{m^2 - p^2 x(1-x) - i\epsilon}}, \tag{2.9}$$

and

$$M_{D=4} = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\,K_0(\sqrt{p \circ p(m^2 - p^2 x(1-x) - i\epsilon)}), \tag{2.10}$$

where K_0 is a modified Bessel function. A crucial point to note is that the α integral is convergent only if $p_E \circ p_E > 0$. For Euclidean momenta this is always true[7]. On the other hand, $p \circ p$ need not be positive definite in Minkowski space when space-time is noncommuting. Specifically, let us choose $\theta^{01} = -\theta^{10} = \Theta_E, \theta^{23} = -\theta^{32} = \Theta_B$ with all other components vanishing. Then

$$p \circ p = \Theta_E^2(p_0^2 - p_1^2) + \Theta_B^2(p_2^2 + p_3^2). \tag{2.11}$$

Therefore, in the case of only space noncommutativity $p \circ p$ is positive definite but for space-time noncommutativity $p \circ p$ can be negative. This fact has very important consequences in the unitarity analysis.

We will now proceed to verify that the generalized unitarity relation (2.1) is satisfied for magnetic theories and violated for electric field theories. First we compute the imaginary part of the Feynman diagram when $p^2 > 0$ and $p \circ p > 0$. It is then easy to show that

$$\text{Im } M_{D=3} = \frac{\lambda^2}{32\pi}\int_{(1-\gamma)/2}^{(1+\gamma)/2} dx\,\frac{\cos(\sqrt{p \circ p}\sqrt{-m^2 + p^2 x(1-x)})}{\sqrt{-m^2 + p^2 x(1-x)}}$$
$$= \frac{\lambda^2}{32\sqrt{p^2}}J_0\left(\frac{\gamma\sqrt{p^2\,p \circ p}}{2}\right), \tag{2.12}$$

---

[7] We will stay away from the region where $p \circ p = 0$ where infrared singularities appear.

where $\gamma = \sqrt{1 - 4m^2/p^2}$.

Using the fact that Im $K_0(-ix) = \frac{\pi}{2} J_0(x)$, where J_0 is a Bessel function, one obtains for $D = 4$ space-time dimensions

$$\text{Im } M_{D=4} = \frac{\lambda^2}{64\pi} \int_{(1-\gamma)/2}^{(1+\gamma)/2} dx\, J_0(\sqrt{p \circ p}\sqrt{-m^2 + p^2 x(1-x)})$$
$$= \frac{\lambda^2}{32\pi} \frac{\sin(\gamma\sqrt{p^2\, p \circ p}/2)}{\sqrt{p^2\, p \circ p}}. \tag{2.13}$$

We will now evaluate the sum over final states (2.6). The integrals evaluate to

$$\sum |M_{D=3}|^2 = \frac{\lambda^2}{4} \frac{1}{8\pi\sqrt{p^2}} \int_0^{2\pi} d\theta\, \cos(p \wedge k) = \frac{\lambda^2}{16\sqrt{p^2}} J_0\left(\frac{\gamma\sqrt{p^2\, p \circ p}}{2}\right) \tag{2.14}$$

and for $D = 4$ space-time (2.6) gives

$$\sum |M_{D=4}|^2 = \frac{\lambda^2}{4} \frac{\gamma}{32\pi^2} \int d\Omega\, \cos(p \wedge k) = \frac{\lambda^2}{16\pi} \frac{\sin(\gamma\sqrt{p^2\, p \circ p}/2)}{\sqrt{p^2\, p \circ p}}. \tag{2.15}$$

We see that for $p \circ p > 0$ the generalized unitarity relation (2.1) is satisfied.

We will now consider the case $p \circ p < 0$. From (2.11) it follows that this configuration of momenta can only exist in the presence of space-time noncommutativity. Moreover p^2 must be negative so it corresponds to space-like momentum. Then

$$\text{Im } M_{D=3} = \frac{\lambda^2}{32\pi} \int_0^1 dx \frac{\sin(\sqrt{|p \circ p|(m^2 + |p^2|x(1-x))})}{\sqrt{m^2 + |p^2|x(1-x))}} \tag{2.16}$$

and

$$\text{Im } M_{D=4} = \frac{\lambda^2}{64\pi} \int_0^1 dx\, J_0(\sqrt{|p \circ p|(m^2 + |p^2|x(1-x))}). \tag{2.17}$$

which are obviously nonzero. However, the right hand side of the equation in fig. 1 is zero because energy-momentum conservation (2.6) forbids a particle with space-like momenta to decay into two massive on-shell particles. Therefore, when $p \circ p < 0$, the generalized unitarity relation (2.1) is violated.

Summarizing, we have shown that field theories with space-time noncomutativity violate the equation in fig. 1 and that field theories with space noncommutativity satisfy it for arbitrary momenta.

2.2. Noncommutative ϕ^4 Scattering Amplitude

Next we consider the $2 \to 2$ scattering amplitude in noncommutative ϕ^4 theory. The Feynman rule for the 4-point vertex in this theory is

$$-i\frac{\lambda}{3}\left(\cos\left(\frac{p_1 \wedge p_2}{2}\right)\cos\left(\frac{p_3 \wedge p_4}{2}\right) + \cos\left(\frac{p_1 \wedge p_3}{2}\right)\cos\left(\frac{p_2 \wedge p_4}{2}\right) + \cos\left(\frac{p_1 \wedge p_4}{2}\right)\cos\left(\frac{p_2 \wedge p_3}{2}\right)\right)$$

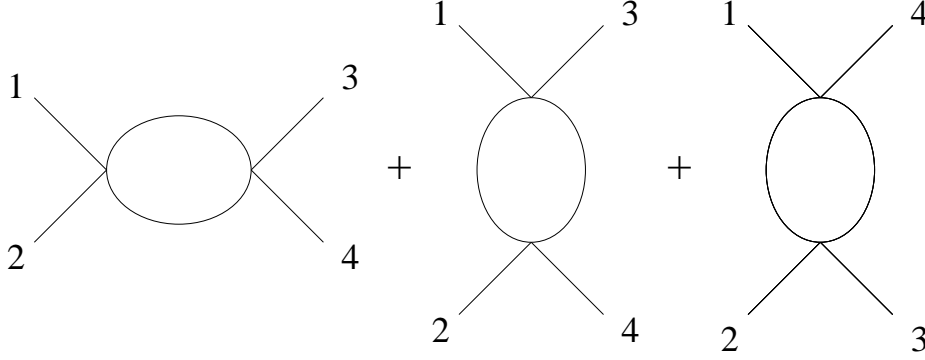where the p_i are momenta entering the vertex.

Fig. 2: One loop diagrams for $2 \to 2$ scattering

The one loop contribution to the $2 \to 2$ scattering amplitude are shown in fig. 2. Evaluating these graphs leads to rather complicated expressions which involve integrals over modified Bessel functions, but these simplify greatly if we expand the expressions in powers of $\theta^{\mu\nu}$. The optical theorem (2.1) for this S-matrix element has to be true term by term in a power series in $\theta^{\mu\nu}$. The leading contribution in θ to the right hand side of (2.1) is the same as commutative ϕ^4 theory, so

$$\sum_n M_{p_1+p_2 \to n} M^*_{n \to p_3+p_4} = \frac{\gamma\lambda^2}{16\pi}\Theta(s - 4m^2) \tag{2.18}$$

where $\Theta(x)$ is a step function, $\gamma = \sqrt{1 - 4m^2/s}$ and $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$.

The leading contribution to the one loop scattering amplitude is

$$M_{p_1+p_2 \to p_3+p_4} =$$
$$-\frac{\lambda^2}{2(4\pi)^2}\left(\int_0^1 dx\left[\ln\left(1 - \frac{s}{m^2}x(1-x)\right) + (s \to t, s \to u)\right] + \frac{2}{3}\ln\left(\frac{m^2}{\mu^2}\right) + \text{const.}\right.$$
$$\left.+ \frac{1}{3}\left(\sum_{i=1}^4 \ln(m^2\, p_i \circ p_i) + \frac{1}{3}\ln(m^2 p_{12} \circ p_{12}) + \ln(m^2 p_{13} \circ p_{13}) + \ln(m^2 p_{14} \circ p_{14})\right)\right) \tag{2.19}$$

8

Here we have defined $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$, $p_{12} = p_1 + p_2$, $p_{13} = p_1 - p_3$, and $p_{14} = p_1 - p_4$. The first line in (2.19) includes the contributions present in ordinary ϕ^4 theory, while the second includes logarithms that are unique to the noncommutative theory. The first logarithm has an imaginary piece when $s > 4m^2$, corresponding to threshold production of two ϕ particles, which gives the precise contribution so that (2.1) is satisfied to leading order in $\theta^{\mu\nu}$ with the right hand side of (2.1) given by formula (2.18). The noncommutative logarithms $\ln(m^2 \, p_i \circ p_i)$ and $\ln(m^2 \, p_{12} \circ p_{12})$ do not contribute an imaginary piece because p_i, p_{12} are time-like, and hence $p_i \circ p_i$, $p_{12} \circ p_{12}$ are positive definite. However, p_{13} and p_{14} are space-like and therefore $p_{13} \circ p_{13}$ and $p_{14} \circ p_{14}$ can be negative if there is space-time noncommutativity. In this case these logarithms have imaginary parts which violate the unitarity relation (2.1) . Therefore, theories with space-time noncommutativity do not have a unitary S-matrix. Moreover, since $p \circ p >$ is always positive for space noncommutative theories there are no new imaginary parts and the optical theorem is satisfied.

3. Discussion

In this paper we have investigated the unitarity of noncommutative scalar field theories. The results we have obtained have a natural interpretation in string theory. We have shown that field theories with space noncommutativity appear to have perturbatively unitarity S-matrix elements and satisfy the generalized unitarity relations of field theory Green's functions. On the other hand, theories with space-time noncommutativity do not have a unitary S-matrix and do not satisfy the cutting rules for Feynman diagrams.

We have done calculations for noncommutative scalar field theories. Even though we have not checked the unitarity of noncommutative gauge theories we have strong reasons to believe that the same results still hold, that is the magnetic theories are unitary while the electric theories are not. This is because the structure of Feynman integrals is the same in gauge and scalar theories. Both have oscillating phases in loop integrations. After analytically continuing momenta and θ^{0i} to Euclidean spacetime, then performing loop integrals one encounters integrals of the form: [8]

$$A \sim \int d\alpha \; \alpha^{1-D/2} \exp^{-\frac{p \circ p}{\alpha}}, \tag{3.1}$$

---

[8] We are not including integration over Feynman parameters which are not needed for the argument.

where p denotes some external momentum. In the Euclidean theory $p \circ p \geq 0$ so that $1/p \circ p$ regulates (3.1) and acts like an ultraviolet cutoff which renders the integral finite. In the theory with only space noncommutativity, the Minkowski expression for $p \circ p > 0$ is never negative. The only possible singularity arises when $\theta^{\mu\nu} p_\nu = 0$ which leads to characteristic infrared singularities of noncommutative field theories [1]-[5]. On the other hand when $\theta^{0i} \neq 0$, the Minkowski expression for $p \circ p$ can be positive or negative so that when one analytically continues the Euclidean answer, one finds branch cuts in the Feynman diagrams for Minkowski $p \circ p < 0$. It is the presence of these extra branch cuts in the loop diagrams of field theories with space-time noncommutativity that are responsible for the failure of the cutting rules and lack of a unitary S-matrix.

The fact that the magnetic gauge theories are unitary can be easily understood from string theory. They provide the appropiate effective description of a low energy limit of string theory in the presence of a background magnetic field [8][9][10]. In this limit, all the massive open string states and the closed strings decouple and the relevant degrees of freedom for the description of the dynamics are the massless open strings. One can build up the effective action for these modes from string theory and show that they are described by noncommutative field theory. Therefore, we expect the field theory to be unitary since string theory in this limit can be appropriately described in terms of noncommutative field theory, without the need of adding any further degrees of freedom. This is indeed what we have found from our field theory analysis.

Field theories with space-time noncommutativity should appear from studying string theory in the presence of a background electric field. This follows from constructing the effective action of open strings in this vacuum. One might expect, based on analogy with a background magnetic field, that there is a similar limit of the string dynamics which is described just by the electric field theory. However, electric fields behave differently than magnetic fields in that they lead to pair production of strings and these destabilize the vacuum if the background electric field exceeds the upper critical value E_c [26][27][28]. Consider for simplicity the electric field to be aligned in the x^1 direction and the metric to be diagonal in the x^0, x^1 plane with each metric component given by g. Reality of the brane action requires that the background electric field on the brane satisfy

$$E \leq E_c, \qquad \text{where} \quad E_c = \frac{g}{2\pi\alpha'}. \tag{3.2}$$

The open strings see a diagonal metric along the x^0, x^1 plane given by G and a noncommutative parameter $\theta^{0i} = \theta$. In terms of the metric g and the background electric field, these parameters are related by the following formula [20][21][22]

$$\alpha' G^{-1} = \frac{1}{2\pi} \frac{E}{E_c} \theta.$$ (3.3)

In order to obtain a field theory of only the massless modes one has to go to the point particle limit $\alpha' \to 0$. From formula (3.3) this implies that, for finite G, that the noncommativity parameter must vanish. Therefore, if one wants a truncation of the full string theory to the theory of only the massless open string modes this can be done but the description of these modes is given by conventional field theory and not noncommutative field theory. Thus, we expect the conventional field theory description to be unitary and it is. Clearly, in order to have a finite noncommutativity parameter θ, α' must be kept finite. This is a string theory and not a field theory. Moreover, since $\theta \sim \alpha'$ there is no scattering process in this string theory which is accurately described only by noncommutative field theory. For scattering processes involving massless open strings of characteristic energy $E \ll \theta^{-1/2}$ conventional field theory is a proper description. Noncommutative field theory becomes relevant for energies of the order of $E \sim \theta^{-1/2}$. However, since $\theta \sim \alpha'$, the energy scale where noncommutativity becomes relevant is precisely the energy scale at which the massive open string states cannot be neglected. Thus, there is no regime in which space-time noncommutative field theory is an appropiate description of string theory. Whenever space-time noncommutative field theory becomes relevant massive open string states cannot be neglected. This gives a very strong indication that noncommutative field theories with space-time noncommutativity are not unitary, since they do not have all the relevant degrees of freedom necessary for an approximate description of a unitary string theory. This is what we found from our field theoretic analysis.

Recently, it has been noticed by several groups [20][21] that it is possible to define a limit of string theory in a background electric field in which the full tower of open string states decouple from the closed strings [20][21]. In this limit the background electric field is sent to its critical value (see [32] for previous analysis of this limit). There is a very simple way of showing that indeed closed strings decouple in this limit. Quantization of open strings with the modified boundary conditions due to the electric field lead to familiar looking mode expansions [25][27] for the light-cone coordinates $X^{\pm}$. In the limit that $E \to E_c$, the waves on the string for the $X^{\pm}$ directions become chiral, that is, they are

either purely right moving or left moving waves. Therefore, in this limit, it is impossible for such an open string to become a closed string, since closed strings require waves which are left moving and right moving.

There is still a lot to learn about noncommutative theories, both field theories with space noncommutativity as well as the recently discovered decoupled open string theories with space time noncommutativity [20][21]. The infrared divergences in the magnetic theories for $p \circ p = 0$ certainly need to better understood within a field theory framework. These theories, as they stand, have no infrared safe observables and the S-matrix is ill-defined. Perhaps nonperturbative input will be required to address this problem.

References

[1] S. Minwalla, M.V. Raamsdonk and N. Seiberg, "Noncommutative Perturbative Dynamics", hep-th/9912072.

[2] M.V. Raamsdonk and N. Seiberg, " Comments on Noncommutative Perturbative Dynamics", hep-th/0002186.

[3] M. Hayakawa,"Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4," hep-th/9912167.

[4] W. Fischler, E. Gorbatov, A. Kashani-Poor, S. Paban, P. Pouliot and J. Gomis, "Evidence for winding states in noncommutative quantum field theory," hep-th/0002067.

[5] A. Matusis, L. Susskind and N. Toumbas, "The IR/UV connection in the noncommutative gauge theories," hep-th/0002075.

[6] N. Seiberg, L. Susskind and N. Toumbas, "The Teleological Behavior of Rigid Regge Rods", hep-th/0005015.

[7] M. Chaichian, A. Demichev and P. Presnajder, "Quantum Field Theory on Noncommutative Space-Times and the Persistence of Ultraviolet Divergences", Nucl. Phys. **B567** (2000) 360, hep-th/9812180; "Quantum Field Theory on the Noncommutative Plane with $E_q(2)$ Symmetry", J. Math. Phys. **41** (2000) 1647, hep-th/9904132.

[8] A. Connes, M.R. Douglas and A. Schwarz, "Noncommutative Geometry and Matrix Theory: Compactification on Tori", JHEP **9802**(1998) 003, hep-th/9711162.

[9] M.R. Douglas and C. Hull, "D-branes and the Noncommutative Torus", JHEP **9802** (1998) 008, hep-th/9711165.

[10] N. Seiberg and E. Witten, " String Theory and Noncommutative Geometry", JHEP **9909** (1999) 032, hep-th/9908142.

[11] O. Andreev and H. Dorn, "Diagrams of Noncommutative ϕ^3 Theory from String Theory, hep-th/003113.

[12] A. Bilal, C.-S. Chu and R. Russo, "String Theory and Noncommutative Field Theories at One Loop", hep-th/003180.

[13] J. Gomis, M. Kleban, T. Mehen, M. Rangamani and S. Shenker, "Noncommutative Gauge Dynamics From The String Worldsheet, hep-th/0003215.

[14] H. Liu and J. Michelson, "Stretched Strings in Noncommutative Field Theory", hep-th/0004013.

[15] A. Bilal, C.-S. Chu, R. Russo and S. Sciuto, "Multiloop String Amplitudes with B-Field and Noncommutative QFT", hep-th/0004183.

[16] Y. Kiem and S. Lee, "UV/IR Mixing in Noncommutative Field Theory via Open String Loops", hep-th/0003145.

[17] A. Rajaraman and M. Rozali, "Noncommutative Gauge Theory, Divergences and Closed Strings", JHEP 0004:033 (2000).

[18] A. Strominger, "Massless Black Holes and Conifolds in String Theory", Nucl. Phys. **B451** (1995) 96, hep-th/9504090.

[19] B.R. Greene, D.R. Morrison and A. Strominger, "Black Hole Condensation and the Unification of String Vacua", Nucl. Phys. **B451** (1995) 109, hep-th/9504145.

[20] N. Seiberg, L. Susskind and N. Toumbas, "Strings in Background Electric Field, Space/TimeNoncommutativity and ANew Noncritical String Theory",hep-th/0005040.

[21] R. Gopakumar, J. Maldacena, S. Minwalla and A. Strominger, "S-Duality and Non-commutative Gauge Theory", hep-th/0005048.

[22] J.L.F Barbón and E. Rabinovici, "Stringy Fuzziness as the Custodial of Time-Space Noncommutativity", hep-th/0005073.

[23] O.J. Ganor, G. Rajesh and S. Sethi, "Duality and Non-Commutative Gauge Theory", hep-th/0005046.

[24] E.S. Fradkin and A.A. Tseytlin, "Nonlinear Electrodynamics From Quantized Strings", Phys. Lett. bf 163B (1985) 123.

[25] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, "String Loop Corrections To Beta Functions", Nucl. Phys. **B288** (1987) 525; A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, "Open Strings in Background Gauge Fields", Nucl. Phys. **B280** (1987) 599.

[26] C.P. Burgess, "Open String Instability in Background Electric Fields", Nucl. Phys. **B294** (1987) 427.

[27] C. Bachas and M. Porrati, "Pair Creation of Open Strings in an Electric Field", Phys. Lett. **B296** (1992) 11.

[28] V.V. Nesterenko, "The Dynamics of Open Strings in a background Electromagnetic Field", Int. J. Mod. Phys. **A4** (1989) 2627.

[29] A. Gonzalez-Arroyo and M. Okawa, "The Twisted Eguchi-Kawai Model:A Reduced Model For Large N Lattice Gauge Theory", Phys. Rev. **D27** (1983)2397.

[30] T. Eguchi and R. Nakayama, "Simplification of Quenching Procedure For Large N Spin Models", Phys. Lett. **B122** (1983) 59.

[31] T. Filk, "Divergences in a Field Theory on Quantum Space", Phys. Lett. **B376** (1996) 53.

[32] S. Gukov, I.R. Klebanov and A.M. Polyakov, " Dynamics of (n, 1) Strings", Phys. Lett. **B423** (1998) 64, hep-th/9711112.